The Cash Accumulation Equation

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1 Deriving the equation

The cash accumulation equation is an equation which tells us how much money we will have in a bank account, at any point in time. The account pays interest, and we are feeding a steady trickle of money into it.

1.1 Compound interest

We will approach the development of this equation by first considering the simpler case, that of just placing a lump sum in an account and then making no additions to the sum. With the usual notation, namely

$$\begin{array}{rcl} y & = & \text{the current sum (dollars)} \\ P & = & \text{principal (dollars)} \\ i & = & \text{force of interest (per year)} \\ t & = & \text{time (years)} \end{array}$$

the equation is

$$y = Pe^{it} \tag{1}$$

and so the sum of money grows exponentially. Differentiating this we derive

$$\frac{dy}{dt} = iPe^{it} \tag{2}$$

and eliminating Pe^{it} between eqns (1) and (2) yields

$$dy = iy\,dt \tag{3}$$

That is to say, eqn (1) is a solution of eqn (3).

1.2 Cash infeed

Having achieved this we are ready to start feeding money into the account, at a rate of F dollars/year. This is effected by making a small change to eqn (3) as follows

$$dy = iy\,dt + F\,dt \tag{4}$$

and accordingly we need to solve the equation

$$t = \int \frac{dy}{iy + F} \tag{5}$$

From a table of integrals, the solution is

$$t = \frac{1}{i}\ln(iy + F) + k \tag{6}$$

where k is the constant of integration. The initial sum deposited was P so we know one point on the curve :

$$(t, y) = (0, P)$$

and making this substitution we find that

$$k = -\frac{1}{i}\ln(iP + F) \tag{7}$$

Using this expression for k, and recalling that

$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$$

gives us the solution :

$$it = \ln\left(\frac{iy + F}{iP + F}\right) \tag{8}$$

This is the neatest form of the cash accumulation equation, as we are calling it, but it not the most useful form. Using the exponential instead of the logarithmic function, the equation can be written out like this :

$$y = Pe^{it} + \frac{F}{i}(e^{it} - 1)\ ,\ i \neq 0 \tag{9}$$

1.3 First special case

From this new perspective, eqn (1) is just a special case of eqn (9) - namely with $F = 0$.

1.4 Second special case

For completeness we will consider the case $i = 0$, and specifically the expression

$$\frac{e^{it} - 1}{i} \ , \ i = 0 \tag{10}$$

One way of evaluating this is to write out the Maclaurin expansion

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \cdots$$

At a glance we can subtract 1 from this series and divide by i, to find out that

$$\frac{e^{it} - 1}{i} = t \ , \ i = 0 \tag{11}$$

With this result the cash accumulation equation now reads

$$y = P + Ft \ , \ i = 0 \tag{12}$$

Thus the cash sum just increases linearly, as expected, if no interest is being paid.

1.5 Third special case

The only other special case to mention is $F = -iP$. Upon making this substitution, eqn (9) becomes simply

$$y = P$$

Evidentally F is negative, and money is being withdrawn rather than deposited. Specifically, the interest is being withdrawn as fast as it is being earned. An alternative interpretation of this special case is that P is negative - the account is overdrawn - and money is being fed in at a rate which just meets the interest charges. A force of interest value is always positive. The next section will consider a case of a perpetually overdrawn account.

2 An application

The cash accumulation equation will now be applied to a problem of optimizing batch size, and we will find it convenient to consider a numeric example. Picture a merchant who deals in a single line of goods. For each item that he buys, he must pay 3 dollars. Regardless of order size he must additionally pay a delivery charge of 8 dollars. The sales volume is 100 items per year.

How many items should he buy, each time he places an order? As a simplification he is not aiming to make a profit, but to charge his customers the lowest breakeven price. We will apply a force of interest of 0.1 per year. In summary:

$$\begin{aligned}
\text{unit purchase price} &= \text{3 dollars} \\
\text{delivery charge} &= \text{8 dollars} \\
\text{sales volume} &= \text{100 per year} \\
\text{force of interest} &= \text{0.1 per year}
\end{aligned}$$

Let N be the number of items that he orders. His bank account, initially with a zero balance, is now overdrawn and stands at $-(3N+8)$ dollars. This happens at the moment called $t = 0$. The N items that he has ordered will last $N/100$ years. Throughout this period he is selling the items, feeding money into the account and reducing the overdraft. The account will momentarily rise to zero, at $t = N/100$, but he must then place another order. We can thus identify two points on his cash accumulation curve :

$$(t, y) = (0, -(3N + 8))$$

and

$$(t, y) = (N/100, 0)$$

After substituting these values, and the value of i, in eqn (9) it looks like this

$$0 = -(3N + 8)\exp(\frac{0.1N}{100}) + \frac{F}{0.1}\left(\exp(\frac{0.1N}{100}) - 1\right) \tag{13}$$

If this relation beween F and N is plotted then it is found to be a curve which passes through a minimum F value. The minimum is roughly at the point

$$(N, F) = (72, 322)$$

So on this basis he should order 72 items - about 9 months' supply - to minimize his resale price and yet break even. If he orders more than 72 then he pays more interest,and if less than 72 then he is hit by the more-frequent delivery charges. Incidentally this lowest value of F, namely 322 dollars/year, reflects a sales volume of 100 items/year. At a glance, therefore, the lowest breakeven unit sale price is 3.22 dollars - that is, 0.22 dollars more than the unit purchase price.

3 Finding the optimum solution

Fortunately we do not need to plot any graphs, to locate the curve minimum. There are two other ways, both using Microsoft Excel. One way is to use

Excel Solver, and the other is to differentiate and then use the Excel Goal Seek module. In the case of a real-world decision one would probably use both, as a check for typing errors. Each method entails typing the right hand side of an equation into an Excel cell. Be sure to include the equals sign, which is the first thing to type.

3.1 Using Excel Solver

If Solver is not initially available from the Excel Tools menu then it needs to be loaded. This is done by selecting

Tools>Add-Ins...>Solver

Eqn (13) can be re-arranged so as to make F the subject of the equation:

$$F = \frac{0.1(3N+8)\exp(0.1N/100)}{\exp(0.1N/100) - 1} \tag{14}$$

The r.h.s. of this re-arranged equation is then typed into a cell. Some other cell needs to be named N, and then Solver is invoked and instructed to minimize the value of F. It remains to admit that the Solver module is difficult, returning values which are sensitive to the initial value of - in this case - N.

3.2 Using Excel Goal Seek

Differentiating eqn (13), and then setting

$$\frac{dF}{dN} = 0$$

generates the equation

$$0 = -(3N+8) + \frac{300}{0.1}\left(\exp(\frac{0.1N}{100}) - 1\right) \tag{15}$$

Excel Goal Seek (which again is in the Tools menu) requires a single-variable equation that has a numeric value on its left hand side, so eqn (15) needs no re-arrangement. Goal Seek is easy and reliable, but only gives us a value for N. If the minimum F value is needed as well, then it is simplest to use the equation

$$F = 0.1(3N+8) + 300 \ , \ \frac{dF}{dN} = 0 \tag{16}$$

which is an intermediate result that arises whilst deriving eqn (15).

4 In conclusion

We think of a batch-size calculation as being used by someone, such as a book publisher, who lays out a lump sum and then gets the money back in dribs and drabs.
As another example, picture a factory machine which is used for producing a succession of batches and for which there is a significant set-up cost associated with each batch-change. In this case the start-of-batch cost is independant of batch size, and our $(3N + 8)$ becomes simply (8). But the calculation follows the same procedure.
Or think of a builder who is becoming ever more heavily overdrawn, whilst his building is going up, with the account making a sharp return to zero when the building is sold. Again, the cash accumulation equation is applicable.